$\boxed{\text{Ex}}$ $$\int_0^{\infty} \frac{1}{\sqrt{x^2+4}}\,dx = \lim_{a\to\infty} \int_{x=0}^{x=a} \frac{1}{\sqrt{x^2+4}}\,dx$$

let $x = 2\tan\theta$ $\qquad x^2+4 = 4\tan^2\theta + 4$

$dx = 2\sec^2\theta\,d\theta \qquad = 4\sec^2\theta$

$x = a$
$x = 2\tan\theta$
$\frac{a}{2} = \tan\theta$
$\theta = \arctan\left(\frac{a}{2}\right)$

$$= \lim_{a\to\infty} \int_0^{\arctan(a/2)} \frac{1}{2\sec\theta}\, 2\sec^2\theta\,d\theta$$

$$= \lim_{a\to\infty} \int_0^{\arctan\left(\frac{a}{2}\right)} \sec\theta\,d\theta$$

$$\int \sec\theta\,d\theta = \int \frac{(\sec\theta)(\sec\theta+\tan\theta)}{(\sec\theta+\tan\theta)}\,d\theta$$

$u = \sec\theta + \tan\theta$
$du = (\sec\theta\tan\theta + \sec^2\theta)\,d\theta$

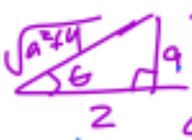
$$= \int \frac{du}{u}$$

$$= \ln|u| + C$$

$$= \ln|\sec\theta + \tan\theta| + C.$$

$$= \lim_{a\to\infty} \ln|\sec\theta + \tan\theta| \Big|_0^{\arctan\left(\frac{a}{2}\right)}$$

$$= \lim_{a\to\infty} \Big(\ln\Big| \underbrace{\sec\left(\arctan\left(\tfrac{a}{2}\right)\right)}_{\theta} + \frac{a}{2} \Big| - \underbrace{\ln|1+0|}_{0} \Big)$$



$\tan\theta = \frac{a}{2}$

$\sec\theta = \frac{\sqrt{a^2+4}}{2} = \sec\left(\arctan\left(\frac{a}{2}\right)\right)$

$$= \lim_{a\to\infty} \Big(\ln\Big| \underset{\downarrow\,\infty}{\frac{\sqrt{a^2+4}}{2}} + \underset{\downarrow\,\infty}{\frac{a}{2}} \Big| - 0 \Big| = \boxed{\infty}.$$

Ex $\lim_{x\to 0^+} \underbrace{x^3}_{\downarrow\, 0} \underbrace{\ln(x)}_{\downarrow\, -\infty} = \lim_{x\to 0^+} \dfrac{\ln(x)}{\frac{1}{x^3}}$ $\quad -\dfrac{\infty}{\infty}$

$\overset{l'H}{=} \lim_{x\to 0^+} \dfrac{\frac{1}{x}}{-\frac{3}{x^4}} = \lim_{x\to 0^+} \dfrac{x^3}{-3}$

$= \boxed{0}.$

Ex $\lim_{y\to 0^+} \left(\dfrac{1+\sin(y)}{1-\sin(y)}\right)^{\frac{1}{y}+2} = ?$

1^{∞} indeterminate

Let $L = \lim_{y\to 0^+} \left(\dfrac{1+\sin(y)}{1-\sin(y)}\right)^{\frac{1}{y}+2}$

$\Rightarrow \ln(L) = \ln\left(\lim_{y\to 0^+}\left(\dfrac{1+\sin(y)}{1-\sin(y)}\right)^{\frac{1}{y}+2}\right)$

since $\ln(x)$ is continuous

$= \lim_{y\to 0^+} \ln\left(\left(\dfrac{1+\sin(y)}{1-\sin(y)}\right)^{\frac{1}{y}+2}\right)$

$\ln L = \lim_{y\to 0^+} \underbrace{\left(\frac{1}{y}+2\right)}_{+\infty} \underbrace{\ln\Big(\underbrace{\frac{1+\sin(y)}{1-\sin(y)}}_{1}\Big)}_{0}$

$$= \lim_{y \to 0^+} \frac{\ln\left(\frac{1+\sin(y)}{1-\sin(y)}\right)}{\underbrace{\frac{y \cdot 1}{y\left(\frac{1}{y}+2\right)}}_{\frac{y}{1+2y}}} \qquad \frac{0}{0} \text{ form.}$$

$$= \lim_{y \to 0^+} \frac{\ln(1+\sin(y)) - \ln(1-\sin(y))}{\frac{y}{1+2y}}$$

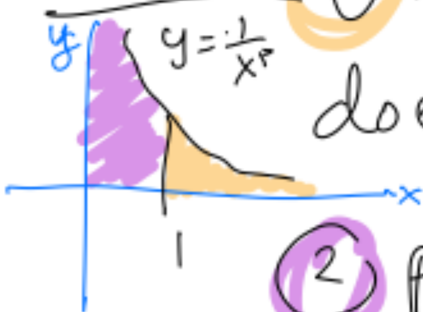
$$\overset{l'H}{=} \lim_{y \to 0^+} \frac{\frac{\cos(y)}{1+\sin(y)} - \frac{-\cos(y)}{1-\sin(y)}}{\frac{(1+2y)1 - y(2)}{(1+2y)^2}}$$

$$= \lim_{y \to 0^+} \frac{\underbrace{\frac{\cos(y)}{1+\sin(y)}}_{\to 1} + \underbrace{\frac{\cos(y)}{1-\sin(y)}}_{\to 1}}{\underbrace{\frac{1}{(1+2y)^2}}_{\to 1}}$$

$$\ln L = 2$$

$$\Rightarrow L = e^{\ln L} = \boxed{e^2}$$

Question: (1) For what positive numbers p



does $\displaystyle\int_1^\infty \frac{1}{x^p}\,dx$ converge?

(2) For what positive numbers p

does $\displaystyle\int_0^1 \frac{1}{x^p}\,dx$ converge

["Converge" — the limit exists – i.e. finite area.]

Solutions: (1) $\displaystyle\int_1^\infty \frac{1}{x^p}\,dx = \lim_{b\to\infty}\int_1^b \frac{1}{x^p}\,dx$

x^{-p}

$$= \lim_{b\to\infty}\left.\begin{cases} \dfrac{x^{-p+1}}{-p+1} & \text{if } p\neq 1 \\ \ln(b) & \text{if } p=1 \end{cases}\right|_1^b$$

$p>1$: 0 or $p<1$: ∞

$$= \lim_{b\to\infty}\left(\begin{array}{ll} \dfrac{b^{-p+1}}{-p+1} - \dfrac{1}{-p+1} & \text{if } p\neq 1 \\ \underbrace{\ln(b)}_{\to +\infty} - \underbrace{\ln(1)}_{=0} & \text{if } p=1 \end{array}\right)$$

$$\boxed{\int_1^\infty \frac{1}{x^p}\,dx} = \begin{cases} \infty & \text{if } p\leq 1 \text{ diverges} \\ \dfrac{1}{p-1} & \text{if } p>1 \text{ converges.} \end{cases}$$

$-\dfrac{1}{-p+1} = \dfrac{1}{p-1}.$

2

$$\int_0^1 \frac{1}{x^p}\,dx = \lim_{b\to 0^+}\int_b^1 \frac{1}{x^p}\,dx \qquad x^{-p}$$

$$= \lim_{b\to 0^+}\left.\begin{cases}\dfrac{x^{-p+1}}{-p+1} & \text{if } p\neq 1\\ \ln(x) & \text{if } p=1\end{cases}\right|_b^1$$

$$= \lim_{b\to 0^+}\begin{cases}\dfrac{1}{-p+1} - \dfrac{b^{-p+1}}{-p+1} & \text{if } p\neq 1\\ \ln(1) - \ln(b) & \text{if } p=1\end{cases}$$

$b^{-p+1} \to 0$ ($p<1$) or ∞ ($p>1$)

$\ln(b) \to -\infty$ diverges

$$\boxed{\int_0^1 \frac{1}{x^p}\,dx = \begin{cases}\infty & \text{if } p\geq 1\\ \dfrac{1}{-p+1} & \text{if } 0<p<1\end{cases}}$$